THE r-STIRLING NUMBERS << of the second kind >>

Cube array basis on Stirling numbers of the second kind

Mohammad R .Serajian Asl (2017)

"serajianasl@yahoo.com"

The r-Stirling numbers represent a certain generalization of the regular Stirling numbers, which, According to Tweedie [2] were so named by Nielsen [3] in honor of James Stirling, who computed them in 1730. [1]

Introduction:

In this paper we are going to generate the r-Stirling numbers of the second kind via a cube array basis on Stirling numbers triangular array, it means that the up mentioned cub array, in fact is formed of r-Stirling tables in different sections of a numerical cube array

Key words:

r-Stirling numbers of the second kind ; cube array basis on r-Stirling numbers

Srirling numbers of the second kind triangular array

"Figure 1"

$n \setminus k$	1	2	3	4	5	6	7	8
1	1							
2	1	1						
3	1	3	1					
4	1	7	6	1				
5	1	15	25	10	1			
6	1	31	90	65	15	1		
7	1	63	301	350	140	21	1	
8	1	127	966	1701	1050	266	28	1
9	1	255	3025	7770	6951	2646	462	36
10	1							

Shift all of the columns to highest row for creating a square array as down figure

"Figure 2"

$n \setminus k$	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	3	6	10	15	21	28	36
3	1	7	25	65	140	266	462	750
4	1	15	90	350	1050	2646	5880	11880
5	1	31	301	1701	6951	22827	63987	159027
6	1	63	966	7770	42525	179487	627396	1899612
7	1	127	3025	34105	246730	1323652	5715424	20912320
8	1	255	9330	145750	1379400	9321312	49329280	216627840
9	1							

Squared

Squared Stirling numbers of the second kind

The set of the numbers locates in each one of the rows in up created square array makes an arithmetic sequence, for example the sequence created in row No. 3, {n = 3} is as below figure

{1 , 7 , 25 , 65 , 140 , 266 .... 1155 ...}

And each one of the terms is equal with an equation, for example the equations for up mentioned sequence {n = 3} are as down figure

$$....1 = 1\cdot\left(\frac{1^3+1^2}{2}\right)$$

$$....7 = 1\cdot\left(\frac{1^3+1^2}{2}\right)+1\cdot\left(\frac{2^3+2^2}{2}\right)$$

$$..25 = 1\cdot\left(\frac{1^3+1^2}{2}\right)+1\cdot\left(\frac{2^3+2^2}{2}\right)+1\cdot\left(\frac{3^3+3^2}{2}\right)$$

$$..65 = 1\cdot\left(\frac{1^3+1^2}{2}\right)+1\cdot\left(\frac{2^3+2^2}{2}\right)+1\cdot\left(\frac{3^3+3^2}{2}\right)+1\cdot\left(\frac{4^3+4^2}{2}\right)$$

$$140 = 1\cdot\left(\frac{1^3+1^2}{2}\right)+1\cdot\left(\frac{2^3+2^2}{2}\right)+1\cdot\left(\frac{3^3+3^2}{2}\right)+1\cdot\left(\frac{4^3+4^2}{2}\right)+1\cdot\left(\frac{5^3+5^2}{2}\right)$$

$$266 = 1\cdot\left(\frac{1^3+1^2}{2}\right)+1\cdot\left(\frac{2^3+2^2}{2}\right)+1\cdot\left(\frac{3^3+3^2}{2}\right)+1\cdot\left(\frac{4^3+4^2}{2}\right)+1\cdot\left(\frac{5^3+5^2}{2}\right)+1\cdot\left(\frac{6^3+6^2}{2}\right)$$

As we can see the coefficients of above equations make a triangular array as down figure

1							
1	1						
1	1	1					
1	1	1	1				
1	1	1	1	1			
1	1	1	1	1	1		
1	1	1	1	1	1	1	
1	1	1	1	1	1	1	1

For another example the sequence created in row No. 4, {n = 4} of square array is as below figure

1 , 15 , 90 , 350 , 1050 , 2646 .... 5880}

And the equations for up mentioned sequence {n = 4} are as down figure

$$.....1 = ..1 \cdot \left(\frac{1^3 + 1^2}{2}\right)$$

$$....15 = ..3 \cdot \left(\frac{1^3 + 1^2}{2}\right) + ..2 \cdot \left(\frac{2^3 + 2^2}{2}\right)$$

$$....90 = ..6 \cdot \left(\frac{1^3 + 1^2}{2}\right) + ..5 \cdot \left(\frac{2^3 + 2^2}{2}\right) + ..3 \cdot \left(\frac{3^3 + 3^2}{2}\right)$$

$$..350 = 10 \cdot \left(\frac{1^3 + 1^2}{2}\right) + ..9 \cdot \left(\frac{2^3 + 2^2}{2}\right) + ..7 \cdot \left(\frac{3^3 + 3^2}{2}\right) + ..4 \cdot \left(\frac{4^3 + 4^2}{2}\right)$$

$$1050 = 15 \cdot \left(\frac{1^3 + 1^2}{2}\right) + 14 \cdot \left(\frac{2^3 + 2^2}{2}\right) + 12 \cdot \left(\frac{3^3 + 3^2}{2}\right) + ..9 \cdot \left(\frac{4^3 + 4^2}{2}\right) + ..5 \cdot \left(\frac{5^3 + 5^2}{2}\right)$$

$$2646 = 21 \cdot \left(\frac{1^3 + 1^2}{2}\right) + 20 \cdot \left(\frac{2^3 + 2^2}{2}\right) + 18 \cdot \left(\frac{3^3 + 3^2}{2}\right) + 15 \cdot \left(\frac{4^3 + 4^2}{2}\right) + 11 \cdot \left(\frac{5^3 + 5^2}{2}\right) + 6 \cdot \left(\frac{6^3 + 6^2}{2}\right)$$

The coefficients of above equations make a triangular array as down figure

1							
3	2						
6	5	3					
10	9	7	4				
15	14	12	9	5			
21	20	18	15	11	6		
28	27	25	22	18	13	7	
36	35	33	30	26	21	15	8

By shifting the columns of the obtained triangular arrays {created by coefficients} to the highest row of triangular array we will change it to a square array as down table {for a sample of a square array of above mentioned coefficients}

$n \setminus k$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	3	5	7	9	11	13	15	17
3	6	9	12	15	18	21	24	27
4	10	14	18	22	26	30	34	38
5	15	20	25	30	35	40	45	50
6	21	27	33	39	45	51	57	63
7	28	35	42	49	56	63	70	77
8	36	44	52	60	68	76	84	92
9	45							

It means that there is a square array of coefficients for each one of the rows locates in Stirling numbers of the second kind squared array as down samples

$n \setminus k$	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1
4	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1
6	1	1	1	1	1	1	1	1
7	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1
9	1							

Square array of coefficients for the row No.1 {n = 1} of Stirling numbers of the second kind squared array

$n \setminus k$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	3	5	7	9	11	13	15	17
3	6	9	12	15	18	21	24	27
4	10	14	18	22	26	30	34	38
5	15	20	25	30	35	40	45	50
6	21	27	33	39	45	51	57	63
7	28	35	42	49	56	63	70	77
8	36	44	52	60	68	76	84	92
9	45							

Square array of coefficients for the row No.2 {n = 2} of Stirling numbers of the second kind squared array

$n \setminus k$	1	2	3	4	5	6	7	8
1	1	4	9	16	25	36	49	64
2	7	19	37	61	91	127	169	217
3	25	55	97	151	217	295	385	487
4	65	125	205	305	425	565	725	905
5	140	245	380	545	740	965	1220	1505
6	266	434	644	896	1190	1526	1904	2324
7	462	714	1022	1386	1806	2282	2814	3402
8	750	1110	1542	2046	2622	3270	3990	4782
	1155							

Square array of coefficients for the row No.3 {n = 3} of Stirling numbers of the second kind squared array

n \ k	1	2	3	4	5	6	7	8
1	1	8	27	64	125	216	343	512
2	15	65	175	369	671	1105	1695	2465
3	90	285	660	1275	2190	3465	5160	7335
4	350	910	1890	3410	5590	8550	12410	17290
5	1050	2380	4550	7770	12250	18200	25830	35350
6	2646	5418	9702	15834	24150	34986	48678	65562
7	5880	11130	18900	29694	44016	62370	85260	113190
8	11880	21120	34320	52200	75480	104880	141120	184920
9	22275							

Square array of coefficients for the row No.4 {n = 4} of Stirling numbers of the second kind squared array

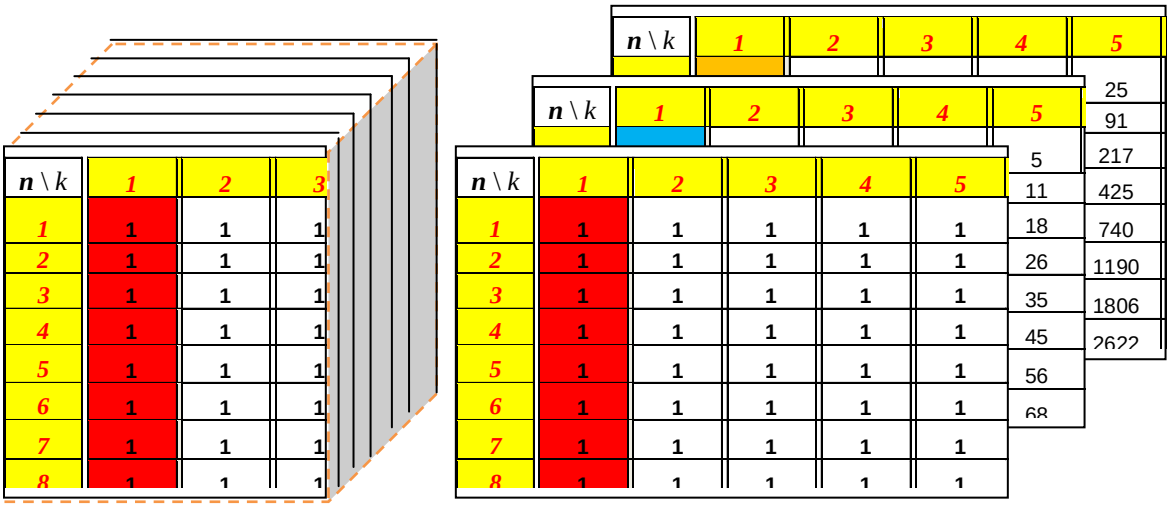
n \ k	1	2	3	4	5	6	7	8
1	1	16	81	256	625	1296	2401	4096
2	31	211	781	2101	4651	9031	15961	26281
3	301	1351	4081	9751	19981	36751	62401	
4	1701	5901	15421	33621	64701	113701		
5	6951	20181	47271	95781	174951			
6	22827	58107	124887	238287				
7	63987	147147	294987					
8	159027	337227						
9	359502							

Square array of coefficients for the row No.5 {n = 5} of Stirling numbers of the second kind squared array

n \ k	1	2	3	4	5	6	7	8
1	1	32	243	1024	3125	7776	16807	32768
2	63	665	3367	11529	31031	70993	144495	
3	966	6069	23772	70035	170898	365001		
4	7770	35574	116298	305382	688506			
5	42525	156660	447195	1071630				
6	179487	563409	1446291					
7	627396	1740585						
8	1899612							
9								

Square array of coefficients for the row No. 6 {n = 6} of Stirling numbers of the second kind squared array No. 8

Then by putting respectively side together the above mentioned Square array of coefficients we will have 3- Dimensional array as a cube, in the name r-Stirling cube array as down figure



Set of squared arrays of coefficients make the 3-D array in the name **r-Stirling cube array**

There are sequences relations between the numbers locates in directions {columns, rows, diagonals} of **r-Stirling cube array**

For example the set of the number locate in **No. 5 {n = 5}** of square array **No.8** make an arithmetic sequence as below

No.8 n=5	42525	156660	447195	1071630				

42525		**156660**		**447195**		**1071630**		**2263065**		**4345320**		**7748055**		**13021890**		**20853525**	
	114135		290535		624435		1191435		2082255		3402735		5273835		7831635		11227335
		176400		333900		567000		890820		1320480		1871100		2557800		3395700	
			157500		233100		323820		429660		550620		686700		837900		1004220
				75600		90720		105840		120960		136080		151200		166320	
					15120		15120		15120		15120		15120		15120		15120

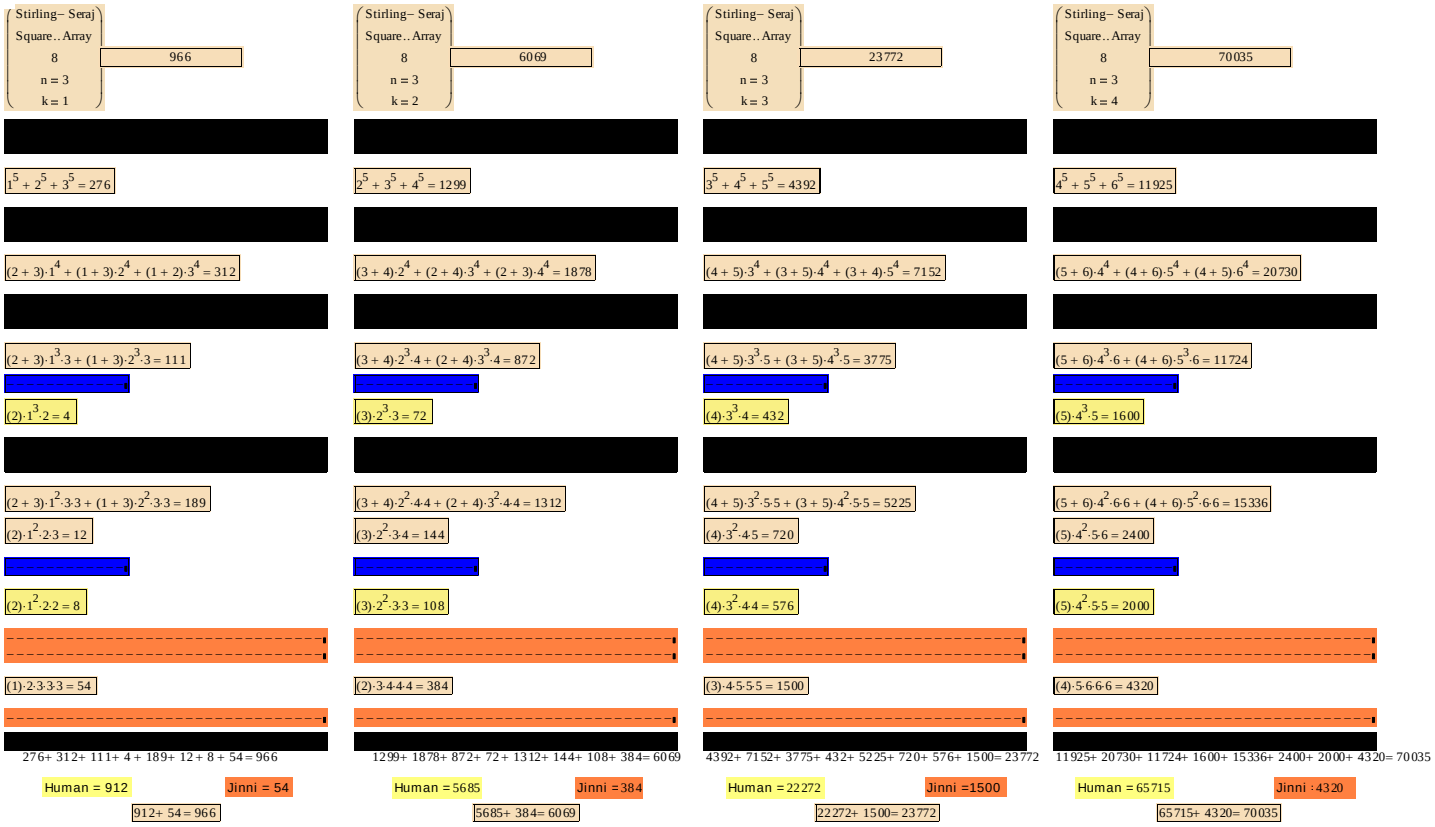
For determining the terms in rows of square array as up mentioned example, there is a way by using of an algorithm in the name family tree of the Human and Jinni for generating Stirling numbers of the second kind

The other example is determining the terms of sequences in row No.3 {n = 3} located in square array No.8 by using of Algorithm in the name family tree of the Human and Jinni for generating Stirling numbers of the second kind, as below

No.8 n=3	966	6069	23772	70035	170898	365001		

By adding one unit to the base numbers we can determine the next term in the sequence as down

For more information about {Algorithm family tree of the Human and Jinni} please refer to the articles in **"mrserajian.ir"** *web site*



A question and an imagination*: is it possible that as squared Stirling numbers of the second kind array mentioned in first of this paper each one of the terms locate in same rows of squared coefficients array be equal with an equation; in that case we can create separated cube array basis on recent coefficients for each one of related rows it means that we can create 4D- array basis on Stirling numbers array in the name* ***r-(r- Stirling numbers the second kind)*** *and perhaps with some related applications for it. this is just a guess and an imagination*

By using of the cube array basis on stirling numbers of the second kind it is possible to compute the (*Tweedie, Nielsen`s*) r-Stirling tables as below in next page

n \ k	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1
4	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1
6	1	1	1	1	1	1	1	1
7	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1
9	1							

r = 1											
	1	r = 2									
		1	1	r = 3							
			1	1	1	r = 4					
				1	1	1	1	r = 5			
					1	1	1	1	1	r = 6	
						1	1	1	1	1	1

$\left\{ {n \atop k} \right\}_1$	k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7
n = 1	1						
n = 2	1	1					
n = 3	1	3	1				
n = 4	1	7	6	1			
n = 5	1	15	25	10	1		
n = 6	1	31	90	65	15	1	
n = 7		63	301	350	140	21	1
n = 8			966	1701	1050	266	
n = 9				7770	6951	2646	
n = 10					42525	22827	
n = 11						179487	

Table 1. r = 1

$\left\{ {n=8 \atop k=5} \right\}_1 = 1050$, $\left\{ {n=7 \atop k=5} \right\}_1 = 140$, $\left\{ {n=7 \atop k=4} \right\}_1 = 350$

$\left\{ {n=8 \atop k=5} \right\}_1 = \left\{ {n=7 \atop k=5} \right\}_1 * k + \left\{ {n=7 \atop k=4} \right\}_1$

n \ k	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	3	5	7	9	11	13	15	17
3	6	9	12	15	18	21	24	27
4	10	14	18	22	26	30	34	38
5	15	20	25	30	35	40	45	50
6	21	27	33	39	45	51	57	63
7	28	35	42	49	56	63	70	77
8	36	44	52	60	68	76	84	92
9	45							

r = 1											
	1	r = 2									
		3	2	r = 3							
			6	5	3	r = 4					
				10	9	7	4	r = 5			
					15	14	12	9	5	r = 6	
						21	20	18	15	11	6

$\left\{ {n \atop k} \right\}_2$	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8
n = 2	1						
n = 3	2	1					
n = 4	4	5	1				
n = 5	8	19	9	1			
n = 6	16	65	55	14	1		
n = 7	32	211	285	125	20	1	
n = 8		665	1351	910	245	27	1
n = 9			6069	5901	2380	434	
n = 10				35574	20181	5418	
n = 11					156660	58107	
n = 12						563409	

Table 2. r = 2

$\left\{ {n=9 \atop k=7} \right\}_2 = 434$, $\left\{ {n=8 \atop k=7} \right\}_2 = 27$, $\left\{ {n=8 \atop k=6} \right\}_2 = 245$

$\left\{ {n=9 \atop k=7} \right\}_2 = \left\{ {n=8 \atop k=7} \right\}_2 * k + \left\{ {n=8 \atop k=6} \right\}_2$

n \ k	1	2	3	4	5	6	7	8
1	1	4	9	16	25	36	49	64
2	7	19	37	61	91	127	169	217
3	25	55	97	151	217	295	385	487
4	65	125	205	305	425	565	725	905
5	140	245	380	545	740	965	1220	1505
6	266	434	644	896	1190	1526	1904	2324
7	462	714	1022	1386	1806	2282	2814	3402
8	750	1110	1542	2046	2622	3270	3990	4782
	1155							

r = 1											
	1	r = 2									
		7	4	r = 3							
			25	19	9	r = 4					
				65	55	37	16	r = 5			
					140	125	97	61	25	r = 6	
						266	245	205	151	91	36

$\left\{ {n \atop k} \right\}_3$	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8	k = 9
n = 3	1						
n = 4	3	1					
n = 5	9	7	1				
n = 6	27	37	12	1			
n = 7	81	175	97	18	1		
n = 8	243	781	660	205	25	1	
n = 9	$r^{(n-r)}$	3367	4081	1890	380	33	1
n = 10		14197	23772	15421	4550	644	
n = 11				116298	47271	9702	
n = 12					447195	124887	
n = 13						1446291	

Table 3. r = 3

$\left\{ {n=10 \atop k=4} \right\}_3 = 14197$, $\left\{ {n=9 \atop k=4} \right\}_3 = 3367$, $\left\{ {n=9 \atop k=3} \right\}_3 = r^{(n-r)}$

$\left\{ {n=10 \atop k=4} \right\}_3 = \left\{ {n=9 \atop k=4} \right\}_3 * k + \left\{ {n=9 \atop k=3} \right\}_3$

[1] - **THE r-STIRLING NUMBERS* Andrei Z. BRODER 1 ... - Semantic Scholar**
***https://pdfs.semanticscholar.org/cdf8/69131c657b88eea5d766803fb2eea5d05023.pdf*by AZ BRODER - 1984**

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[3] - N. Nielsen, Handbuch der Theorie der Gammafunktion (B.G. Teubner, Leipzig, 1906); reprinted under the title: Die Gammafunktion (Chelsea, New York, 1965).